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Some Absolutely Continuous Operators I

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P. A. Rejto

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ABSTRACT

In this report abstract conditions are formulated which assure that the part of an operator over a given interval is absolutely continuous. These conditions are verified for the operators $-\frac{d^2}{dx^2} + q(x)$ acting on the intervals $(-\infty, +\infty)$ and $[0, +\infty)$ respectively, provided that the function q dies out at infinity.

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1. Introduction

Let A be a strictly self adjoint operator acting in an abstract Hilbert space $\underline{H}$ and let $\{E(\lambda)\}$ be its spectral resolution. Consider the set of those vectors $\{f\}$ in $\underline{H}$ for which the scalar valued function $(f, E(\lambda)f)$ of the variable λ is absolutely continuous. According to Halmos and Kato [1] [17] these vectors form a subspace which reduces the operator A and the restriction of A to this subspace, is called the absolutely continuous part of A. In analogy with this terminology we call the restriction of A to its eigenspace corresponding to a given interval, the part of A over this interval.

In this report we formulate abstract conditions which assure that the part of an operator over a given interval is absolutely continuous. For this purpose it is convenient to write the given operator as the sum of an unperturbed operator and a perturbation. Then in Section 2 we introduce two such sets of conditions. One of the features of these conditions is that the perturbed resolvent does not appear explicitly in them. Nevertheless to establish the first set of these conditions we need information on the perturbed resolvent, as is illustrated in the Appendix. In order to eliminate the use of the perturbed resolvent, we introduce additional conditions, obtaining Conditions 1-2-3. In Section 3 we state an abstract lemma which plays a basic role in establishing this set of conditions for specific operators. With the aid of this lemma, in Sections 4 and 5, we

verify these conditions for the operators $-\frac{d^2}{dx^2} + q(x)$ on the intervals $(-\infty, +\infty)$ and $[0, +\infty)$, respectively. We shall see that the main requirement is that the function q should die out at infinity, as it was emphasized by E. C. Titchmarch [14].

Most likely these conditions could be established for a wider class of operators, e.g. for the operator $-\Delta + q$, Δ being the Laplacian and the function q satisfying the Povsner-Ikebe condition [9], [10]. Another example could be the operator $M + K$, M being the multiplication operator and K satisfying a gentleness condition introduced by Friedrichs [6], [7], or a condition of partial gentleness introduced by the author, elsewhere [8]. At present, however, we are not concerned with these questions.

In conclusion let us remark that C. R. Putnam formulated a condition for the absolute continuity of normal operators. His condition involved the commutator of the given operator and of an operator to be constructed [2], [18]. Let us also remark that the notion of an absolutely continuous operator was extended to operators which may not admit a spectral resolution. Specifically Sz. Nagy [4] assigned to an arbitrary contraction operator a unitary transformation, which in turn, he used to assign an operator valued measure to the contraction. Later M. Schreiber [5] formulated conditions for the absolute continuity of contractions.

2. Conditions for the absolute continuity of the part of an operator over an interval.

We start from the resolvent loop integral formula concerning the strictly self adjoint operator A and its spectral resolution $\{E(\zeta)\}$. According to this formula, [16.b] if $\mathrm{Im}\, z \neq 0$ and $R(z) = (z-A)^{-1}$, then for every f in $\underline{H}$

$$(2.1)\quad (f,E(\zeta_2)f) - (f,E(\zeta_1)f) = \frac{1}{2\pi i} \lim_{\varepsilon \to +0} \int_{\zeta_1}^{\zeta_2} \left[(f,R(\zeta-i\varepsilon)f) - (f,R(\zeta+i\varepsilon)f)\right] d\zeta$$

provided that ζ_1 and ζ_2 are not point-eigenvalues of A. Note that some authors define the resolvent to be $(A-z)^{-1}$ and accordingly there is an additional minus sign in the equation. Suppose that for some dense set of vectors $\underline{S}$, the numerically valued functions of the variable ζ in the right member of (2.1), converge to an integrable function as ε converges to 0; moreover the order of integrating and taking limits can be interchanged. Then clearly the part of A over the interval $[\zeta_1,\zeta_2]$ is absolutely continuous.

In particular, this is the case, if for each element f of S, the two families of scalar valued functions $(f,R(\zeta\pm i\varepsilon)f)$ converge, as ε converges to 0 from above, uniformly in ζ in $[\zeta_1,\zeta_2]$. Thus the absolute continuity of the part of A over this interval is assured by this property, which we formulate for future reference.

[*] There is a dense set S such that for every ζ in $[\zeta_1, \zeta_2]$ the two families of Hermitian forms, $R(\zeta \pm i\epsilon)$ on $S \times S$, can be continued weakly to the point ζ. Moreover this is uniform in ζ in $[\zeta_1, \zeta_2]$.

For convenience set

$$A_1 = A_o + P ,$$

A_o and A_1 being strictly self adjoint on the same domain. For brevity we call A_o and A_1 perturbed and unperturbed operators respectively; P is the perturbation. Distinguishing the corresponding resolvents by subscripts, for non real z we have the elementary relation,

$$R_1(z) - R_o(z) = R_1(z) P R_o(z) ,$$

which is called the second resolvent equation. In case $(1 - PR_o(z))$ admits an inverse this equation allows one to express $R_1(z)$ in terms of $R_o(z)$ and P,

$$R_1(z) = R_o(z)(1 - PR_o(z))^{-1} . \tag{2.2}$$

Next we assume that the unperturbed resolvent has property [*], and ask for conditions on the family of operators $(1 - PR_o(\zeta + i\epsilon))^{-1}$ which ensure that the perturbed resolvent also has property [*]. First one is tempted to say that all that one needs is the strong convergence of this sequence on S, arguing that the product of a strongly and a weakly convergent sequence of

operators converges weakly. Then according to (2.2) the perturbed resolvent would converge weakly. However this argument does not apply to our operators since we do not have convergence on the entire space. Another pathology concerning weak convergence of operators on a dense set is that the space of bounded operators is not complete with reference to this convergence. In order to exclude such pathologies we need some additional conditions and definitions.

Consider an abstract Banach space $\underline{B}$ and a form F on $\underline{B} \times \underline{B}$ which is conjugate linear in its first and linear in its second argument, in short, Hermitian. Let $< \bar{f}, Fg >$ denote the value of F for the pair (f,g). The complex conjugate sign above f does not imply that the conjugate of an element of an abstract Banach space is defined; it merely emphasizes that F is conjugate linear in its first argument. In analogy to the notion of the norm of an operator we define the norm of the form F by setting

$$||<F>|| = \sup_{f,g} \frac{|<\bar{f}Fg>|}{||f||_B \cdot ||g||_B} \quad . \tag{2.3}$$

In analogy to weak convergence of a family of operators, we say that the family of forms $F(\varepsilon)$ converges weakly to the form $F(0)$ if for every f and g in $\underline{B}$

$$\lim_{\varepsilon \to 0} < \bar{f}, F(\varepsilon)g > = < \bar{f}, F(0)g> \quad . \tag{2.4}$$

Note that if $F(\varepsilon)$ is a weak Cauchy sequence and the corresponding norms $||\langle F(\varepsilon)\rangle||$, remain bounded independently of ε then $F(\varepsilon)$ converges to some form, say $F(0)$. Let A be a bounded operator on $\underline{B}$ and define, FA, the product of the form F with the operator A, to be the form determined by

$$(2.5) \qquad \langle \overline{f}, FAg \rangle = \langle \overline{f}, F(Ag) \rangle .$$

Concerning such products we have a simple but important proposition: *if the family of forms* $F(\varepsilon)$ *is bounded in norm and converges weakly to* $F(0)$ *and if the family of operators* $A(\varepsilon)$ *converges strongly to* $A(0)$, *then the family of forms* $F(\varepsilon)A(\varepsilon)$ *converges weakly to the form* $F(0)A(0)$. This an immediate consequence of definitions (2.4) and (2.5) if we remember that for the family of operators $\{A(\varepsilon)\}$ the principle of uniform boundedness holds.

After this digression on abstract Hermitian forms we return to the question of finding conditions which ensure that property [*] holds for the perturbed resolvent. Recall that we already assumed that [*] holds for the unperturbed resolvent and we consider the set $\underline{S}$ entering it. Then roughly speaking we shall require the existence of a norm such that: if $\underline{B}$ is the completion of $\underline{S}$ with reference to this norm then one can set

$$F(\varepsilon) = R_o(\zeta+i\varepsilon)$$

and

$$A(\varepsilon) = \left(1-PR_o(\zeta+i\varepsilon)\right)^{-1},$$

in the previous proposition. Since the operators on the right act on $\underline{H}$ let us specify how they determine forms and operators respectively on $\underline{B}$.

Suppose that there is a Banach space $\underline{B}$ such that $\underline{B} \cap \underline{H}$ is dense in both $\underline{B}$ and $\underline{H}$ in the appropriate norm. Let A be a bounded operator on H which maps $\underline{B} \cap \underline{H}$ into itself and whose restriction to $\underline{B} \cap \underline{H}$ is closeable with reference to the $\underline{B}$-norm. Then we call the closure of this operator the operator determined by A. Similarly let F be a bounded operator on $\underline{H}$ and suppose that the form

$$(f,Fg) \quad \text{on} \quad (\underline{B} \cap \underline{H}) \times (\underline{B} \cap \underline{H})$$

is closeable with reference to the $\underline{B}$-norm. Then we call the closure of this form the form determined by F. Now we can formulate the previously mentioned conditions:

Condition 1. $[\zeta_1,\zeta_2]$

For every ζ *in this interval and* $\varepsilon \neq 0$ *the two families of operators* $R_o(\zeta \pm i\varepsilon)$ *on* $\underline{H}$ *determine forms on* $\underline{B} \times \underline{B}$. *These forms converge weakly, as* ε *converges to* 0 *from above, uniformly in* ζ. *Moreover their norms remain bounded uniformly in* $\varepsilon > 0$ *and in* ζ.

Condition I. $[\zeta_1,\zeta_2]$

For every ζ *in this interval and* $\varepsilon \neq 0$ *the two families of operators* $1 - PR_o(\zeta \pm i\varepsilon)$ *on* $\underline{H}$, *determine invertible operators on* $\underline{B}$. *The two families of inverses,* $(1-PR_o(\zeta \pm i\varepsilon))^{-1}$

converge strongly on B as ε converges to 0 from above, and this is uniform in ζ.

We see from the previous proposition and from the second resolvent equation (2.2) that if the perturbation problem A_0, P, satisfies Condition 1-I.$[\zeta_1,\zeta_2]$ then property [*] holds for the perturbed resolvent, $R_1(\zeta\pm i\varepsilon)$. Thus these conditions assure the absolute continuity of the part of A_1 over the interval $[\zeta_1,\zeta_2]$.

In the Appendix we shall verify this condition for the Sturm-Liouville operator $-D^2 + M(q)$, provided that the potential q dies out at infinity. In doing this, we shall need detailed information on the perturbed resolvent, in spite of the fact that it does not enter the condition explicitly. For this reason we introduce two more conditions, which are more restrictive then this one. Nevertheless for a large class of perturbation problems we can verify them without reference to the perturbed resolvent. They read as follows:

Condition 2.$[\zeta_1,\zeta_2]$

For every ζ in this interval the two families of operators determined by $PR_0(\zeta\pm i\varepsilon)$ on B, converge in norm to compact operators, as ε converges to 0 from above, and this is uniform in ζ.

Condition 3.$[\zeta_1,\zeta_2]$

For every ζ in this interval the nullspaces of $1-PR_0(\zeta\pm)$ on B are trivial.

If this condition is satisfied at a given point ζ only then we refer to it as Condition $3.\zeta$. Similarly we shall speak of Condition $1.\zeta$ and $2.\zeta$.

We claim that Conditions 2-3 imply Condition I. For, according to Condition 2, $PR_o(\zeta+)$ is compact and so the Fredholm alternative applies to the operator $1-PR_o(\zeta+)$. Thus from Condition 3 we can conclude that $1-PR_o(\zeta+)$ admits a bounded inverse defined on all of $\underline{B}$. From this in turn, using the other part of Condition 2 we conclude that for small enough ε the operators $1-PR_o(\zeta+i\varepsilon)$ admit inverses; moreover $(1-PR_o(\zeta+i\varepsilon))^{-1}$ converges in the B-operator norm to $(1-PR_o(\zeta+))^{-1}$. Thus Conditions 2 and 3 imply Condition I. In conclusion let us remark that if we replace norm convergence by strong convergence in Condition 2 this implication may possibly be false.

Summarizing our results we arrive at the following:

Lemma 2.1

Suppose that the perturbation problem A_o,P, satisfies conditions $1\text{-}I.[\zeta_1,\zeta_2]$ or Conditions $1\text{-}2\text{-}3.[\zeta_1,\zeta_2]$. Then the part of the perturbed operator $A_o + P$ over the interval $[\zeta_1,\zeta_2]$ is absolutely continuous.

In the applications that we consider we can verify the first two of these conditions directly. In establishing Condition $3.[\zeta_1,\zeta_2]$, however, it is convenient to proceed indirectly, i.e., to assume that it is violated and to derive a false conclusion from this. Such a conclusion is stated in the next section.

3. The basic lemma

Let V be a spectral transformation of the unperturbed operator A_o, i.e.

$$A_o = V^* M V , \tag{3.1}$$

M being the multiplication operator. For convenience assume that the spectrum of A_o is simple, then M acts on an $\underline{L_2}$-space of numerically valued functions. Let $\underline{C}$ be the space of continuous functions with the supremum norm and assume that V is a bounded map from $\underline{H} \cap \underline{B}$ into $\underline{C}$. Finally let δ_ζ denote the evaluation functional on $\underline{C}$ defined by

$$\delta_\zeta f = f(\zeta) . \tag{3.2}$$

Then we formulate a lemma which plays a basic role in establishing Condition 3.ζ.

Lemma 3.1

Suppose that the perturbation problem A_o, P *satisfies Conditions* 1-2.$[\zeta_1,\zeta_2]$, *and suppose that the vector* f *in* B *and number* ζ *in* $[\zeta_1,\zeta_2]$ *are such that*

$$\pm \text{ either } (1-PR_o(\zeta-))f = 0 \text{ or } (1-PR_o(\zeta-))f = 0. \tag{3.3}$$

Then

$$\delta_\zeta V f = 0 \tag{3.4}$$

We start the proof by setting

$$g(\varepsilon) = PR_o(\zeta+i\varepsilon)f$$

and observing that according to Condition 2.ζ this family of functions converges for an arbitrary function f in $\underline{B}$. If we take f to be the function appearing in assumption $(3.3)_+$ then clearly

$$g(\varepsilon) \longrightarrow f$$

and

$$(3.5) \qquad (1-PR_o(\zeta+))g(\varepsilon) \longrightarrow 0 .$$

From this fact and from Conditions 1.ζ we conclude that

$$(3.6) \qquad \lim_{\varepsilon \to +o} < \overline{f} , R_o(\zeta-i\varepsilon)(1-PR_o(\zeta+))g(\varepsilon) > = 0 .$$

Next we claim that

$$(3.7) \qquad \mathrm{Im} < \overline{f} , R_o(\zeta-i\varepsilon)g(\varepsilon) > = 0 .$$

For, the form $R_o(\zeta-i\varepsilon)$ on $\underline{B} \times \underline{B}$ is determined by the operator $R_o(\zeta-i\varepsilon)$ on H; hence

$$< \overline{f} , R_o(\zeta-i\varepsilon)g(\varepsilon) > = \lim_n < \overline{f}_n , R_o(\zeta-i\varepsilon)g_n(\varepsilon) > ;$$

f_n and g_n being sequences in $\underline{B} \cap \underline{H}$ which tend to f and g respectively in $\underline{B}$. By assumption $PR_o(\zeta+i\varepsilon)$ is a bounded map of $\underline{B} \cap \underline{H}$ into itself and so we can take

$$g_n(\varepsilon) = PR_o(\zeta+i\varepsilon)f_n .$$

Thus

$$< \overline{f} , R_o(\zeta-i\varepsilon)g(\varepsilon) > = \lim_n (R_o(\zeta+i\varepsilon)f_n , PR_o(\zeta+i\varepsilon)f_n) ,$$

and since P on $\underline{H}$ is self-adjoint we see the validity of (3.7). From relations (3.7) and (3.5) we conclude that

$$\lim_{\varepsilon \to +o} \left[\mathrm{Im} < \overline{f} , R_o(\zeta - i\varepsilon)(1 - PR_o(\zeta+)) g_\varepsilon > \right] = < \overline{f} , R_o(\zeta-) f > .$$

On the other hand according to (3.6) the left member equals zero, thus

$$\mathrm{Im} < \overline{f} , R_o(\zeta-) f > = 0 . \tag{3.8}$$

Now from this relation we can easily derive conclusion (3.4). For, according to Condition 1.ζ, for an arbitrary function f in $\underline{B}$,

$$< \overline{f} , R_o(\zeta-) f > = \lim_{\varepsilon, n} (f_n, R_o(\zeta - i\varepsilon) f_n) , \tag{3.9}$$

the limit on the right being a double limit. Next set

$$S_o(z) = (z - M)^{-1}$$

and observe that according to (3.1)

$$R_o(z) = V^* S_o(z) V .$$

Hence

$$\mathrm{Im}(f_n , R_o(\zeta - i\varepsilon) f_n) = (Vf_n , \left[S_o(\zeta - i\varepsilon) - S(\zeta + i\varepsilon) \right] Vf_n) .$$

From this fact and from the assumption that V maps vectors in $\underline{B} \cap \underline{H}$ into continuous functions we can conclude

$$\lim_{\varepsilon \to +o} (Vf_n , \left[S_o(\zeta - i\varepsilon) - S_o(\zeta + i\varepsilon) \right] Vf_n) = \pi |\delta_\zeta Vf_n|^2 ,$$

if we use a theorem of Fatou. Since V is a bounded map of $\underline{B}$ $\underline{H}$ into $\underline{C}$, this yields

$$(3.9) \qquad \lim_{\varepsilon,n} \left[\mathrm{Im}(f_n, R_o(\zeta - i\varepsilon) f_n) \right] = \pi |\delta_\zeta V f|^2$$

Finally combining relations (3.8), (3.9) and (3.10 we obtain (3.4), which establishes the lemma.

4. The operator $-\frac{d^2}{dx^2} + q(x)$ on $(-\infty, +\infty)$

Let $\underline{D}$ be the set of twice continuously differentiable functions on $[-\infty, +\infty]$ with bounded support. We define the unperturbed operator to be the $\underline{D}$-closure of the formal operator $-\frac{d^2}{dx^2}$ and denote it by $-D^2$. For the perturbation we take the operator defined by

$$M(q)f(x) = q(x)f(x) . \tag{4.1}$$

As is well known the operator $-D^2$ is strictly self adjoint. Hence if the function q is bounded so is the operator $M(q)$ and we see that $-D^2 + M(q)$ is also strictly self adjoint and we take it to be the perturbed operator.

We formulate additional conditions on the function q which assure that the perturbation problem $-D^2$, $M(q)$, satisfies the abstract conditions of Section 2. Specifically we assume that q is a continuous function for which

$$\lim_{|x| \to \infty} q(x) = 0 . \tag{4.2}$$

Moreover, q is integrable with respect to a weight function,

$$\int_{-\infty}^{\infty} |q(x)| w(x) dx < \infty , \tag{4.3}$$

such that $w(x) \geq 1$ and

$$\lim_{|x| \to \infty} w(x) = \infty . \tag{4.4}$$

Define the Banach space $\underline{B}$ to be the closure of $\underline{D}$ with reference to the norm

$$(4.5) \qquad ||f||_B = \sup_x |f(x)| + \int_{-\infty}^{\infty} |f(x)|w(x)dx \ .$$

Note that assumptions (4.3) and (4.4) imply that

$$(4.6) \qquad ||q||_B < \infty \ .$$

We maintain that the perturbation problem $-D^2$, $M(q)$ satisfies the conditions of Section 2. More specifically we maintain the following:

Subtheorem 4.1

Suppose that the function q satisfies conditions (4.3), (4.4) and that the bounded interval $[\zeta_1,\zeta_2]$ does not contain the point 0 or a point eigenvalue of the operator $-D^2 + M(q)$ in L_2. Then the perturbation problem $-D^2$, $M(q)$ satisfies conditions 1-2-3'$[\zeta_1,\zeta_2]$ with reference to the Banach space B.

The proof of this subtheorem makes essential use of the representation of the unperturbed resolvent. Specifically setting

$$(4.7) \qquad k(u) = e^{i|u|} \ ,$$

we have

$$(4.8) \qquad R_o(z)(x,y) = \frac{1}{\sqrt{z}} k(\sqrt{z}(x-y)) \ ,$$

$\sqrt{z}$ being defined in the plane cut along the positive real axis and having positive imaginary part.

a) Condition 1.$[\zeta_1,\zeta_2]$. Let $[\zeta_1,\zeta_2]$ be a closed and bounded interval not containing the point 0. First observe that for

ζ in $[\zeta_1, \zeta_2]$ the kernel of the unperturbed resolvent converges pointwise. Specifically if on the positive real axis

$$\sqrt{\zeta} = \lim_{\epsilon \to +0} \sqrt{\zeta + i\epsilon} \ ,$$

then according to (4.7) ,

$$\lim_{\epsilon \to +0} R_o(\zeta + i\epsilon)(x, y) = \frac{1}{\sqrt{\zeta}} k(\sqrt{\zeta}(x - y)) \ . \tag{4.9}$$

Since this convergence is uniform in any bounded (x, y) region, we see that setting

$$R_o(\zeta+)(x, y) = \frac{1}{\sqrt{\zeta}} k(\sqrt{\zeta}(x - y)) \ , \tag{4.10}$$

for any pair of functions (f, g) in $\underline{D}$,

$$\lim_{\epsilon \to +0} < \bar{f}, R_o(\zeta + i\epsilon) g > = < \bar{f}, R_o(\zeta+) g > . \tag{4.11}$$

Note that $R_o(\zeta+)$ is a possibly unbounded Hermitian form, defined by the kernel (4.10) on $\underline{D} \times \underline{D}$. We claim that it is bounded uniformly in ζ . For, we see from relations (4.7) and (4.8) that

$$R_o(\zeta + i\epsilon)(x, y) = O(1)$$

uniformly in x , y , ϵ and ζ in $[\zeta_1, \zeta_2]$. This implies

$$|< \bar{f}, R_o(\zeta + i\epsilon) g > | = O(1) \| f \|_B \cdot \| g \|_B \tag{4.12}$$

if we remember that according to definition (4.5) the $\underline{B}$-norm majorizes the $\underline{L}_1$-norm. Thus the family of Hermitian forms $R_o(\zeta + i\epsilon)$ on $\underline{B} \times \underline{B}$ remains bounded in norm and converges weakly on the dense set $\underline{D} \times \underline{D}$. Since $[\zeta_1, \zeta_2]$ is a closed interval not containing 0 , relation (4.11) is uniform in ζ in $[\zeta_1, \zeta_2]$ too. This establishes the part of condition 1 . $[\zeta_1, \zeta_2]$ concerning the family of operators $R_o(\zeta + i\epsilon)$. A repetition of these arguments shows that the family of operators $R_o(\zeta - i\epsilon)$ also

satisfies this condition. Thus the validity of the entire Condition 1. $[\zeta_1, \zeta_2]$ is established.

b. Condition 2. $[\zeta_1, \zeta_2]$. Similarly to the previous subsection we first claim that the kernel of the family of operators $M(g)R_o(\zeta + i\epsilon)$ converges pointwise. For, we see from (4.7) that

$$\lim_{\epsilon \to +0} M(g)R_o(\zeta + i\epsilon)(x,y) = \frac{1}{\sqrt{\zeta}} g(x)k(\sqrt{\zeta}(x-y)) .$$

Relations (4.6) and (4.7) show that this kernel defines a bounded operator on $\underline{B}$ and we set

$$M(g)R_o(\zeta+)(x,y) = \frac{1}{\sqrt{\zeta}} g(x)k(\sqrt{\zeta}(x-y)) . \tag{4.13}$$

It is nearly evident from these relations that $M(g)R_o(\zeta + i\epsilon)$ converges strongly to $M(g)R_o(\zeta+)$. We maintain that this convergence holds even in the operator norm, i.e.,

$$\lim_{\epsilon \to 0} \| M(g)R_o(\zeta + i\epsilon) - M(g)R_o(\zeta+) \|_B = 0 , \tag{4.14}$$

moreover it is uniform in ζ in $[\zeta_1, \zeta_2]$.

To see this we approximate the operator $M(q)R_o(\zeta+i\varepsilon)$ by a double sequence. Specifically let

$$d_m(x) = \begin{cases} 1 & 0 \leq |x| \leq m \\ m+1-|x| & m \leq |x| \leq m+1 \\ 0 & m+1 \leq |x| \end{cases}$$

and set

$$A(m,n,\varepsilon) = M(d_m)M(q)R_o(\zeta+i\varepsilon)M(c_n) , \tag{4.15}$$

c_n being the characteristic function of the interval $[-n,n]$. These cut off kernels behave better than the original kernel, inasmuch as they have bounded support. Hence for each (m,n) the limit

$$\lim_{\varepsilon \to +o} A(m,n,\varepsilon)(x,y)$$

exists uniformly over the entire (x,y)-space. Thus setting

$$A(m,n,+) = M(d_m)M(q)R_o(\zeta+)M(c_n) ,$$

we have

$$\lim_{\varepsilon \to +o} ||A(m,n,\varepsilon) - A(m,n,+)||_B = 0 . \tag{4.16}$$

Next we claim that setting

$$A(\infty,n,\varepsilon) = M(q)R_o(\zeta+i\varepsilon)M(c_n) ,$$

we have

$$\lim_{m \to \infty} ||A(\infty,n,\varepsilon) - A(m,n,\varepsilon)||_B = 0 , \tag{4.17}$$

uniformly in n and $\varepsilon \neq 0$. This is nearly evident from the uniform boundedness estimate (4.11) and from conditions (4.2), (4.3) and (4.4) which imply that

$$\lim_{m \to \infty} ||d_m q - q||_B = 0 .$$

Finally we claim that setting

$$A(m,\infty,\varepsilon) = M(d_m)M(q)R_o(\zeta+i\varepsilon) ,$$

we have

$$\lim_{n \to \infty} ||A(m,\infty,\varepsilon) - A(m,n,\varepsilon)||_B = 0 , \tag{4.18}$$

uniformly in m and $\varepsilon \neq 0$. For, evidently

$$A(m,\infty,\varepsilon) - A(m,n,\varepsilon) = M(d_m)M(q)R_o(\zeta+i\varepsilon)M(1-c_n)$$

and using the uniform boundedness estimate (4.11) again,

$$||M(d_m)M(q)R_o(\zeta+i\varepsilon)M(1-c_n)f|| = O(1)||(1-c_n)f||_{L_1} .$$

Since the weight function w appearing in definition (4.5) satisfies condition (4.4), we see that

$$\lim_{n \to \infty} \frac{||(1-c_n)f||_{L_1}}{||f||_B} = 0 .$$

Insertion of these estimates in relation (4.18) establishes the relation. Combining relations (4.16), (4.17) and (4.18) we see that the triple limit of the sequence $A(m,n,\varepsilon)$ exists and hence all the iterated limits are equal.

Thus

$$(4.19)\qquad \lim_{\varepsilon\to+o}\left(\lim_{m,n} A(m,n,\varepsilon)\right) = \lim_{m,n}\left(\lim_{\varepsilon\to+o} A(m,n,\varepsilon)\right)$$

and (4.14) follows. It also follows that this limit is uniform in the variable $\zeta \in [\zeta_1,\zeta_2]$, which we suppressed in equation (4.15).

It remains to be shown that the operator $M(q)R_o(\zeta+)$ is compact. To see this observe that each of the operators $A(m,n,\varepsilon)$ is compact. For, definition (4.15) shows that this operator maps an arbitrary bounded set of functions in $\underline{B}$ into a set of equicontinuous functions, which are pointwise uniformly bounded. Since the support of the image functions is contained in the interval $[-m-1,m+1]$, the Arzela-Ascoli compactness criterion applies. This establishes the compact character of the operator $M(q)R_o(\zeta+)$.

c) Condition 3.ζ. We maintain that if the point $\zeta \neq 0$ is not an eigenvalue of the operator $-D^2 + M(q)$ in $\underline{L}_2$ then this condition holds. We shall derive this fact from a lemma that follows, which in turn will be derived from the basic Lemma 3.1. Recall that we denoted by $\underline{C}$ the space of continuous functions with the supremum norm.

Lemma 4.2

Suppose that the point $\zeta \neq 0$ and the function $f \neq 0$ in $\underline{B}$ are such that, either

$$(4.20)_+\qquad (1-M(q)R_o(\zeta+)f = 0\ ,$$

or

(4.20)$_-$ $\qquad (1-M(q)R_o(\zeta-))f = 0 \ .$

Then there is a function $g \neq 0$ in C such that

(4.21) $\qquad \lim_{|x|\to\infty} g(x) = 0 \ ,$

and g is a weak solution of

(4.22) $\qquad (-D^2+M(q))g = \zeta g \ ,$

with reference to the set of test functions D.

Since ζ is in the continuous spectrum of $-D^2$, the operator $\zeta + D^2$ admits a densely defined inverse, $R_o(\zeta)$ in L_2. Let us assume, for a moment, that the function f of relation (4.20)$_\pm$ is in $(\zeta+D^2)D \subset L_2$ and hence it is in the domain of $R_o(\zeta)$. That is we assume that there is a function g in D, such that

(4.23) $\qquad f = (\zeta+D^2)g \quad , \quad g = R_o(\zeta)f \ .$

Then the first of these equations evidently implies

(4.24) $\qquad (1-M(q)R_o(\zeta))f = (\zeta+D^2)g - M(q)g$

since

$$R_o(\zeta)(\zeta+D^2) = 1 \quad \text{on} \quad D \ .$$

Hence if the function f satisfies the additional equation

$$(1-M(q)R_o(\zeta))f = (1-M(q)R_o(\zeta\pm))f \ ,$$

then (4.24) yields conclusion (4.22) in the strong sense.

We maintain that even if the function f is not in $(\zeta+D^2)\underline{D}$, this argument can be modified to yield conclusion (4.22) in the weak sense. To see this, recall relation (4.14) which shows that the family of functions $R_o(\zeta+i\varepsilon)f(x)$ is uniformly bounded over the entire x-interval $[-\infty, +\infty]$; moreover

$$\lim_{\varepsilon \to +o} R_o(\zeta+i\varepsilon)f(x) = R_o(\zeta+)f(x)$$

holds uniformly over any bounded interval. Thus setting

$$(4.25) \qquad g(x) = R_o(\zeta+)f(x) = \frac{1}{\sqrt{\zeta}} \int_{-\infty}^{\infty} k\big(\sqrt{\zeta}(x-y)\big)f(y)dy \ ,$$

we obtain a function in $\underline{C}$. This relation is a limiting case of the previous relation (4.23). Instead of specifying the limit procedure involved we prefer to verify from definition (4.25) that this function satisfies the conclusions of the lemma. From the assumption that f is in $\underline{B}$, hence it is integrable and continuous, we see that the function g is twice continuously differentiable, moreover

$$(4.26)_x \qquad \frac{d^2}{dx^2} g(x) + \zeta g(x) = f(x) \ .$$

Let $\underline{D}^+$ be the set of twice continuously differentiable functions in $\underline{C}$. According to definition (4.25) g is in $\underline{D}^+$ and setting

$$\partial^2 g(x) = \frac{d^2}{dx^2} g(x)$$

we have

(4.26) $$\partial^2 g + \zeta g = f$$

We note that the operators $D^2 + \zeta$ on $\underline{D}$ and $\partial^2 + \zeta$ on $\underline{D}^+$ are formal adjoints to each other. Hence for every test function t in $\underline{D}$,

$$< g, (D^2+\zeta)t > = < (\partial^2+\zeta)g, t > ,$$

which in view of relation (4.27) yields

$$< g, (D^2+\zeta)t > = < f, t > .$$

Remembering definition (4.23) and assumption $(4.20)_{\pm}$ we see that

$$f = M(q)g$$

which inserted in the previous relation yields,

$$< g, (D^2+\zeta)t > = < M(q)g, t > .$$

Hence

$$< g, (M(q)-D^2)t > = < g, \zeta t > ,$$

establishing (4.22) in the weak sense.

To complete the proof of the lemma we have to establish relation (4.21). We claim that this relation is a simple consequence of the basic Lemma 3.1. For, an elementary argument shows that a spectral transformation of the unperturbed operator $-D^2$ is given by the vector valued kernel,

$$V(x,y) = \frac{1}{\sqrt{2}\,\pi}\, x^{-1/4} \begin{pmatrix} e^{i\sqrt{x}\,y} \\ e^{-i\sqrt{x}\,y} \end{pmatrix} .$$

Thus using a slightly extended version of Lemma 3.1 we have

$$Vf(\zeta) = 0 ,$$

that is

$$\int_{-\infty}^{\infty} e^{i\sqrt{\zeta}\, y} f(y)dy = \int_{-\infty}^{\infty} e^{-i\sqrt{\zeta}\, y} f(y)dy = 0 .$$

Insertion of this relation in definition (4.25) shows the validity of conclusion (4.21). This completes the proof of Lemma 4.1.

Having established the lemma we derive our original statement namely:

Corollary 4.3

Suppose that the point $\zeta \neq 0$ is not a point eigenvalue of the operator $-D^2 + M(q)$ in L_2. Then the perturbation problem $-D^2$, $M(q)$, satisfies Condition 3.ζ.

First let us consider the case of a strictly positive number ζ. Since the function q is integrable, we know that equation (4.22),

$$(-D^2+M(q))g = \zeta g ,$$

admits two linearly independent solutions in C which oscillate at plus and minus infinity [16.e], Hence every non-trivial solution of this equation oscillates at plus and minus infinity. Thus conclusion (4.21) of Lemma 4.1 is contradicted and the lemma shows that the nullspaces of $1-M(q)R_o(\zeta\pm)$ on B are trivial. In other words for strictly positive numbers ζ Condition 3.ζ is established.

Next let us consider the case of a strictly negative ζ and assume that Condition 3.ζ does not hold. Then according to Lemma 4.1, for such a value of ζ equation (4.22) admits a non-trivial solution g which vanishes at plus and minus infinity. An elementary argument shows, which we shall not carry out, that such a function g vanishes exponentially at plus and minus infinity. Thus g is square integrable and according to equation (4.22) it is an eigenfunction of the operator $-D^2 + M(q)$ in L_2. This establishes Corollary 4.3.

Finally combining this corollary with the statements of subsections a) and b) we see the validity of Subtheorem 4.1.

Having established the subtheorem we derive two theorems. The one that follows is immediate if we remember the abstract Lemma 2.1.

Theorem 4.1

Suppose that the function q satisfies conditions (4.3), (4.4) and that the interval $[\zeta_1,\zeta_2]$ does not contain the point 0 or a point eigenvalue of the operator $-D^2 + M(q)$ in L_2. Then the part of this operator over the interval $[\zeta_1,\zeta_2]$ is absolutely continuous.

We maintain that this theorem can be generalized inasmuch as the entire continuous part of $-D^2 + M(q)$ is absolutely continuous. For, we have seen that for an arbitrary positive integer the interval $[\frac{1}{n},n]$ does not contain point eigenvalues of $-D^2 + M(q)$. Hence according to this theorem the part of the operator over

this interval is absolutely continuous. It is a general operator theoretic fact that the spectral projectors of a strictly self adjoint operator define a projector valued measure, which is countably additive in the strong topology [16.0]. Thus the part of the operator $-D^2 + M(q)$ over the open interval $(0,\infty)$ is absolutely continuous. On the other hand conditions (4.3) and (4.4) imply that the function q is square integrable. This, in turn, according to a theorem of E. Balslev and F. Wolf [13] implies that $M(q)$ is compact with respect to $-D^2$. Hence Weyl's theorem shows that the essential spectrum of the operator $-D^2 + M(q)$ consists of the interval $[0,\infty]$. Combining this fact with the statement above we arrive at the following:

Theorem 4.2

Suppose that the function q satisfies conditions (4.3) and (4.4). Then the continuous part of the operator $-D^2 + M(q)$ is absolutely continuous.

5. The operator $-\frac{d^2}{dx^2} + q(x)$ on $[0,\infty]$

As is well known in order to define a strictly self adjoint operator with the aid of this formal differential operator one needs a boundary condition. Accordingly let $\underline{D}$ be the set of those twice continuously differentiable functions with bounded support which satisfy the boundary condition,

$$(5.1)_\alpha \qquad \cos\alpha\, f(0) + \sin\alpha\, f'(0) = 0 \ .$$

We define the unperturbed operator to be the $\underline{D}$-closure of the formal operator $-\frac{d^2}{dx^2}$ and denote it by $-D^2$. Similarly we define the perturbation by setting

$$(5.2) \qquad M(q)f(x) = q(x)f(x) \ .$$

Then we know that the perturbed operator $-D^2 + M(q)$ is essentially self adjoint on $\underline{D}$ for a large class of functions q[15].

E. C. Titchmarsch [14], [16.g] formulated a condition on the function q, continuity and integrability, which assured that the continuous part of $-D^2 + M(q)$ was absolutely continuous. His arguments were based on the Weyl representation theorem [16.c] for the perturbed resolvent and on the Titchmarsh - Kodaira formula [16.d], which in turn is based on the resolvent loop integral formula. In the Appendix we show that his estimates imply the abstract Condition I and thus the absolute continuity follows from the loop integral formula alone.

In this section we illustrate that imposing additional conditions on the function q, the perturbation problem $-D^2, M(q)$ satisfies the abstract Conditions 1-2-3, concerning the unperturbed resolvent. Accordingly following the previous section we assume that q is a real valued continuous function for which

$$\lim_{x \to \infty} q(x) = 0 . \tag{5.3}$$

Moreover it is integrable with respect to some weight function w, such that

$$w(x) \geq 1 \quad \text{and} \quad \lim_{x \to \infty} w(x) = \infty . \tag{5.4}$$

In analogy with the previous subsection, define the Banach space $\underline{B}$ to be the closure of $\underline{D}$ with reference to the norm

$$||f||_B = \sup |f(x)| + \int_0^\infty |f(x)| w(x) dx . \tag{5.5}$$

We maintain that conditions (5.3) and (5.4) assure that the perturbation problem $-D^2$, $M(q)$ satisfies Conditions 1-2-3.$[\zeta_1, \zeta_2]$ with respect to the space $\underline{B}$, provided that the interval $[\zeta_1, \zeta_2]$ does not contain 0 or a point eigenvalue of $-D^2 + M(q)$.

The main difference between the operator $-D^2$ acting in free space or in a space of functions satisfying a boundary condition, is that the latter is not translation invariant. Accordingly the inverse can not be represented as a convolution. Nevertheless, if we review the estimates of the previous section we see

that the convolution character of the unperturbed resolvent played but a small role. There were two key properties: one was the uniform boundedness estimate (4.11) and the other was the uniform convergence of the kernels $R_o(\zeta+i\varepsilon)(x,y)$ over any bounded (x,y)-region. We claim that these properties hold for the present unperturbed resolvent as well. For convenience set $\alpha = 0$ in $(5.1)_\alpha$, i.e. consider the case of the boundary condition $f(0) = 0$. As is well known, $R_o(z)$ is an integral operator, with kernel,

$$(5.6) \qquad R_o(z)(x,y) = \begin{cases} \frac{1}{\sqrt{z}} e^{ix\sqrt{z}} \sin(y\sqrt{z}) & y \le x \\ \frac{1}{\sqrt{z}} \sin(x\sqrt{z})\, e^{iy\sqrt{z}} & y \ge x , \end{cases}$$

$\sqrt{z}$ being defined in the plane cut along the positive axis and having positive imaginary part. This formula shows that the kernel of the unperturbed resolvent converges pointwise. More specifically if we define the kernel $R_o(\zeta+)(x,y)$ by replacing $\sqrt{z}$ in (5.6) by

$$\sqrt{\zeta} = \lim_{\varepsilon \to +o} \sqrt{\zeta+i\varepsilon}$$

then we see that

$$(5.7) \qquad \lim_{\varepsilon \to +o} R_o(\zeta+i\varepsilon)(x,y) = R_o(\zeta+)(x,y) ,$$

uniformly in any bounded (x,y)-region. To see the uniform boundedness of these kernels set

$$K_1(\zeta+i\varepsilon)(x,y) = \frac{1}{\sqrt{\zeta+i\varepsilon}}\, C_x(y) e^{ix\sqrt{\zeta+i\varepsilon}} \sin\,(y\sqrt{\zeta+i\varepsilon})$$

(5.8)

$$K_2(\zeta+i\varepsilon)(x,y) = \frac{1}{\sqrt{\zeta+i\varepsilon}}\, (1-C_x(y) e^{iy\sqrt{\zeta+i\varepsilon}} \sin\,(x\sqrt{\zeta+i\varepsilon}) \ .$$

By definition $\mathrm{Im}\sqrt{\zeta+i\varepsilon} \geq 0$ and combining this with the fact that x and y are positive we see that

$$|K_1(\zeta+i\varepsilon)(x,y)| \leq \frac{1}{\sqrt{\zeta+i\varepsilon}}$$

(5.9)

$$|K_2(\zeta+i\varepsilon)(x,y)| \leq \frac{1}{\sqrt{\zeta+i\varepsilon}} \ .$$

Since evidently

$$R_o(\zeta+i\varepsilon)(x,y) = K_1(\zeta+i\varepsilon)(x,y) + K_2(\zeta+i\varepsilon)(x,y) \ ,$$

estimate (5.9) shows that if the closed interval $[\zeta_1,\zeta_2]$ does not contain the point 0 then

(5.10) $$|R_o(\zeta+i\varepsilon)(x,y)| = O(1) \ ,$$

uniformly in x, y, ε, and $\zeta \in [\zeta_1,\zeta_2]$.

Having established relations (5.7) and (5.10) the arguments of the previous section can be repeated, showing the validity of Condition 1-2-3. Thus we arrive at the following:

Theorem 5.1

Suppose that the function q satisfies conditions (5.3) and (5.4). Then the continuous part of the operator $-D^2 + M(q)$ is absolutely continuous.

Appendix

Condition I for the operator $-D^2 + M(q)$ on $[0,\infty)$.

Suppose that the function q satisfies the condition introduced by Titchmarsh, namely, it is continuous and absolutely integrable. Then we show that the perturbation problem $-D^2, M(q)$ satisfies Condition I.$[\zeta_1,\zeta_2]$ with reference to the space L_1, provided that this interval does not contain 0.

We need the Weyl representation theorem [16.c] for the perturbed resolvent which says that for non-real z, $R(z)$ is an integral operator. Moreover

$$R(z)(x,y) = \begin{cases} \phi(x,z)\psi(y,z) & y \leq x \\ \psi(x,z)\phi(y,z) & y \geq x \end{cases},$$

where ϕ and ψ are solutions of the equation

$$-\frac{d^2}{dx^2}\theta(x) + q(x)\theta(x) = z\theta(x),$$

such that ϕ satisfies the boundary condition at $x = 0$, ψ is square integrable at $x = \infty$, and their Wronskian equals 1. Now the estimates of Titchmarsch [14], [16.f] say that

$$\text{(I-1)} \qquad \phi(x,\zeta+i\varepsilon) = O(1), \quad \psi(y,\zeta+i\varepsilon) = O(1)$$

uniformly in x, y, ε, and $\zeta \in [\zeta_1,\zeta_2]$. On the other hand the fact that the solutions of a differential equation depend continuously on the eigenvalue parameter, shows that

$$\text{(I-3)} \qquad \lim_{\varepsilon \to +o} M(q)R_o(\zeta+i\varepsilon) = M(q)R(\zeta+) \ ,$$

strongly on the space L_1 and uniformly in $\zeta \in [\zeta_1,\zeta_2]$. The second resolvent equation shows that

$$(z+D^2)R(z) = 1 + M(q)R_1(z) \ ,$$

which in turn, shows that

$$\text{(I-4)} \qquad \left(1-M(q)R_o(\zeta+i\varepsilon)\right)^{-1} = 1 + M(q)R_1(\zeta+i\varepsilon) \ .$$

Thus relations (I-3) and (I-4) establish Condition I.

Actually for this operator it isn't any easier to establish Condition I, than the fact that the perturbed resolvent can be continued weakly onto the spectrum on $L_1 \supset L_1$.

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a) Theorem X.2.1 and Corollary X.2.3
b) Theorem XII.2.10
c) Theorem XIII.3.16
d) Theorem XIII.5.18
e) Lemma XIII.6.18
f) Exercise XIII.9.H.4 p. 1564
g) Exercise XIII.9.H.7 p. 1566

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